DEVELOPMENT OF BOUNDARY ELEMENT SOFTWARE FOR CONTACT STRESS PROBLEMS - INTERFERENCE PIN WITH FRICTIONAL INTERFACE IN ORTHOTROPIC PLATE AND SUBJECTED TO PLATE LOAD

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Abstract Boundary Element solutions are developed for orthotropic plates with interference pin and subjected to plate loading. The interface between pin and the hole is considered to be infinite or finite. This is a typical contact stress problem and it has non-linear moving boundary with mixed boundary conditions. Inverse formulation is used to obtain displacements, stresses, and loads for a specified separation. The results are compared with Finite Element and Continuulm results available in the literature for validation of the Boundary Element Method Software (ACBMS) being developed.

Keywords: BEM, Contact stress, Inverse formulation, Pinjoints.

INTRODUCTION

Most of contact stress problems are non-linear in nature since contact area changes with applied load. This moving boundary value problem can be solved by Continuum method, FEM or BEM.Continuum method though accurate can be used to solve simple problems. FEM can be used for the analysis of realistic problems involving complex geometries and material properties, but it consumes a lot of data preparation time and computer memory, as the entire domain is to be discretized. BEM effectively overcomes these limitations as the discretisation is limited to the boundary and thus emerges as a powerful tool particularly in the analysis of contact stress problems. With this motivation, the authors are developing a BEM software ACBMS (**A**nalysis of **C**ontact stress problems using **B**oundary Element **M**ethod **S**oftware).

Fastener joints are an important engineering application, which come under the category of contact stress problems. These joints being unavoidable in structures requiring periodic assembly and disassembly are the potential locations for failure due to stress concentration. Thus, they have received considerable attention [3-10]. The literature available on pin joints using Continuum [8-10] and FEM [3,6,7] are plenty, though not all are listed here. But using BEM it is relatively less [4-5]. Lin et.al [4-1993] and Mahajerin et.al [5-1986] have analysed pin joints using BEM, but either they have considered a push-fit problem which is linear or solved mis-fit problem by iterative procedure. Inverse formulation can be effectively used for misfit pin joints which are non linear in nature. In this paper, using inverse formulation, a rigid interference fit pin in an orthotropic plate with frictional interface and subjected to plate load is considered for validation of software. Direct BEM using the fundamental solution given in Lin et.al [4] for orthotropic domain is coded in C language.

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INVERSE TECHNIQUE FOR CONTACT STRESS PROBLEMS

The numerical modeling of the contact stress problems is very difficult because the contact area between the contacting bodies is not known apriori. There are three general techniques to solve such problems.

1. Iterative Method
2. Inverse Method
3. Mixed Method

In the present work, Inverse technique "where in causative load required for a given contact/separation is determined" is used. This method was first adopted for pin joints by Rao and his co-workers at IISc., [6-10]. The solution for analysis lies in getting zero radial pressure at the transition point between the contact and separation regions. This can be achieved by solving a given contact/separation configuration subjected to two different load levels (two load vectors) and the load required to cause the given configuration is obtained by linearly interpolating for zero radial stress at the transition point. This is possible due to linearity of the solution between any two-load levels. Using the above said procedure the software developed is checked for the problem of rigid pin in an orthotropic plate and subjected to plate load and having frictional interface.

PROCEDURE INVOLVED IN BEM FOR SOLVING 2D ELASTO STATIC PROBLEMS

Kelvin's fundamental solution for a point load in an orthotropic material is given by

Ui (p,Q)= Uij (p . Q) . ej

where,

$$U_{XX}(p,Q) = \frac{1}{\beta}\left(\sqrt{\alpha_1}\,A_2^2 \ln(r_1) - \sqrt{\alpha_2}\,A_1^2 \ln(r_2)\right)$$

$$U_{XY}(p,Q) = U_{YX}(p,Q) = -\frac{1}{\beta} A_1 A_2 (\theta_1 - \theta_2) \qquad (1)$$

$$U_{YY}(p,Q) = -\frac{1}{\beta}\left(\frac{1}{\sqrt{\alpha_1}} A_1^2 \ln(r_1) - \frac{1}{\sqrt{\alpha_2}} A_2^2 \ln(r_2)\right)$$

The above functions are called 'Displacement Kernels. The first subscript refers to the direction of displacement of boundary point Q caused by a unit load at the interior or the boundary point P in the direction of second subscript.

In the above functions

$$\beta = 2\pi(\alpha_1 - \alpha_2)S_{22}, \quad A_i = S_{12} - \alpha_i S_{22},$$

$$r_i^2 = x_1^2 + \frac{1}{\alpha_i} x_2^2,$$

$$\theta_i = \arctan\left(x_2 \big/ \sqrt{\alpha_i}\,.\,x_1 \right)$$

and $x_1 = (x_p - x_q), x_2 = (y_p - y_q)$

where α1 and α2 are the roots of the following characteristics equation.

$S_{22}\,\alpha^2 - (2\,S_{12} + S_{66})\,\alpha + S_{11} = 0,$

S_{ij} being the material constants

The traction vector is expressed as :

$t_i = T_{ij}(p,Q).e_J$

where the functions Tij(p,Q) are called 'Traction Kernels' and are defined as

$$T_{xx}(p,Q) = \frac{1}{\beta} x_k n_k \left[\frac{A_1}{\sqrt{\alpha_2 . r_2^2}} - \frac{A_2}{\sqrt{\alpha_1 . r_1^2}} \right]$$

$$T_{xy}(p,Q) = \frac{1}{\beta} \left[\frac{M_1 A_2}{r_1^2} - \frac{M_2 A_1}{r_2^2} \right] \qquad (2)$$

$$T_{yx}(p,Q) = \frac{1}{\beta} \left[\frac{M_1 A_1}{a_1 . r_1^2} - \frac{M_2 A_2}{a_2 . r_2^2} \right]$$

$$T_{yy}(p,Q) = \frac{1}{\beta} x_k n_k \left[\frac{A_1}{\sqrt{\alpha_1 . r_1^2}} - \frac{A_2}{\sqrt{\alpha_2 . r_2^2}} \right]$$

In the above functions

$$M_l = \left(\sqrt{\alpha_i}\, x_1 n_2 - 1 \big/ \sqrt{\alpha_i}\; x_2 n_1 \right) \qquad i = 1,2$$

$$x_k n_k = (x_1 n_1 + x_2 n_2)$$

where n1, n2 are direction cosines.

The displacements, strain and stresses at any point, inside or on the boundary for any other system of loading is obtained by using Betti's reciprocal theorem and the fundamental solution. Betti's reciprocal theorem can be written as

$$\int_s t_i^{(a)} u_i^{(b)} ds = \int_s t_i^{(b)} u_i^{(a)} ds \qquad (3)$$

where the displacement and traction vectors $u_i^{(a)}$ and $t_i^{(a)}$ are unknown to begin with. These $u_i^{(a)}$ and $t_i^{(a)}$ are evaluated by putting the point load at each nodal point.

$u_i^{(b)}$ and $t_i^{(b)}$ are Kelvin solution (fundamental solution) for displacement (1) and traction (2) at any surface point Q due to unit load applied on an interior point p in an infinite domain. The boundary integral equation in terms of kernels is obtained as

$$u_i(p) + \int_s T_{ij}(p,Q).u_j(Q)ds = \int_s U_{ij}(p,Q).t_j(Q)ds \qquad (4)$$

This boundary integral equation forms the basic equation for numerical formulation. For numerical implementation, the boundary curve is divided into N isoparameters elements connected with 2N nodal points. For two dimensional problems, each node is defined by 4 variables ux, uy ,tx ,ty . For a problem with unique solution, two variables are prescribed at every node. Therefore, for 2N unknown variables of the problem, 2N equations are required to solve the problem. Using a fundamental solution, the displacements, and tractions at every node is obtained. This yields are set of linear equation, at node. To produce second set of linear equation the load is applied at node 2 and repeat the use of the fundamental solution to calculate all the variables at the other node. This operation is repeated till the load is placed at the last node which will give final set of equations. Therefore we end up with 2N equation and 2N unknowns which produces a unique solution.

Therefore the boundary integral equation 4 is written as

$$u_i(p) + \sum_1^M \int_e T_{ij}(p,Q) u_j(Q) ds = \sum_1^M \int_e U_{ij}(p,Q) t_j(Q) ds \qquad (5)$$

where M represents the number of elements.

Over each element, the variation of displacement and geometry must be specified. In the present work, isoparametric quadratic elements are adopted. So each element must have three nodes, one at each end and one at the midpoint. The local variable ξ has its origin at the midpoint and values -1 to +1 at the two nodes. The shape functions for these elements are

$$x(\xi) = \sum_{c=1}^{3} N_c(\xi)\ x_c = N_1(\xi)\ x_1 + N_2(\xi)\ x_2 + N_3(\xi)\ x_3$$

$$y(\xi) = \sum_{c=1}^{3} N_c(\xi)\ y_c = N_1(\xi)\ y_1 + N_2(\xi)\ y_2 + N_3(\xi)\ y_3 \qquad (6)$$

$$(u_x)(\xi) = \sum_{c=1}^{3} N_c(\xi)\ (u_x) = N_1(\xi)\ (u_x)_1 + N_2(\xi)\ (u_x)_2 + N_3(\xi)\ (u_x)_3$$

$$(u_y)(\xi) = \sum_{c=1}^{3} N_c(\xi)\ (u_y) = N_1(\xi)\ (u_y)_1 + N_2(\xi)\ (u_y)_2 + N_3(\xi)\ (u_y)_3$$

equation 4 can be written as

$$\begin{bmatrix} C_{xx}(P) C_{xy}(P) \\ C_{yx}(P) C_{yy}(P) \end{bmatrix} \begin{bmatrix} u_x(P) \\ u_y(P) \end{bmatrix} + \sum_{m=1}^{M}\sum_{c=1}^{3}\left(\int_{-1}^{+1} \begin{bmatrix} T_{xx}(P,Q) & T_{xy}(P,Q) \\ T_{yx}(P,Q) & T_{yy}(P,Q) \end{bmatrix} N_c(\xi) J(\xi) d\xi \right) \begin{bmatrix} u_x(Q) \\ u_y(Q) \end{bmatrix}$$

$$= \sum_{m=1}^{M}\sum_{c=1}^{3}\left(\int_{-1}^{+1} \begin{bmatrix} U_{xx}(P,Q) & U_{xy}(P,Q) \\ U_{yx}(P,Q) & U_{yy}(P,Q) \end{bmatrix} N_c(\xi) J(\xi) d\xi \right) \begin{bmatrix} t_x(Q) \\ t_y(Q) \end{bmatrix}$$

This can be written as

$$\begin{bmatrix} C_{xx}(P) C_{xy}(P) \\ C_{yx}(P) C_{yy}(P) \end{bmatrix} \begin{bmatrix} u_x(P) \\ u_y(P) \end{bmatrix} + \sum_{m=1}^{M}\sum_{c=1}^{3} \begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} \begin{bmatrix} u_x(Q) \\ u_y(Q) \end{bmatrix}$$

$$= \sum_{m=1}^{M}\sum_{c=1}^{3} \begin{pmatrix} B_{xx} & B_{xy} \\ B_{yx} & B_{yy} \end{pmatrix} \begin{bmatrix} t_x(Q) \\ t_y(Q) \end{bmatrix}$$

where

$$[A]_{ij} = \int_{-1}^{1} T_{ij}(p,Q) N_c(\xi) J(\xi) d\xi$$

$$[B]_{ij} = \int_{-1}^{1} U_{ij}(p,Q) N_c(\xi) J(\xi) d\xi$$

A set of linear algebraic equations are obtained by taking each node in turn as the load point ‘P’ and performing the integrations as indicated above, equation emerges in the form
[A][u]=[B][t]

The matrices [A] and [B] contain the integrations of the kernels TiJ and UiJ respectively and the parameter cij(P) contributes only to diagonal coefficients of the [A] matrix (when p=Q).

After applying on the boundary conditions, the matrices [A] and [B] are rearranged such that all the known variables are on the right hand side and all the unknown variables are on the left hand side which will result in
[A*][x] = [B*][y]
where [x] contains all unknown variables and [y] contains all the known variables.
Therefore the final system of linear algebraic equation can be written as
[A*][x] = [C]

Gauss elimination technique is used to solve the equation and strain and stress are then obtained from these nodal displacements.

CONFIGURATION AND CONTACT PHENOMENA

A thin, elastic, finite plane with a hole of diameter 2a filled with a rigid pin of diameter 2a(1+λ) is considered, where λ is non-dimensional interference parameter. The positive values of λ represents interference and negative values represent clearance.

When an over - sized rigid pin is inserted in the hole, an interference is developed between the pin and the hole and full contact takes place at their interface with compressive radial stress throughout the contact.

When an uniform load S_x is applied at the edge of the plate, separation initiates at particular load at particular points where compressive stresses reduces to zero and further increase in load gradually makes the plate to separate from pin. Before the separation, the problem is linear with stationary boundary conditions. But once the separation initiates and spreads, the problem becomes non-linear mixed boundary problem. Separation / contact phenomenon is shown in fig.1. In the region of contact the tangential displacement will be zero for infinite friction.

With finite friction , the slip along the pin plate interface is characterized by relative displacements between originally adjoining points on the pin and plate. If the interfacial shear stress is smaller than the friction co-efficient times the normal stress (i.e. $|\sigma_{r\theta}| < \mu\ |\sigma_r|$) there can be no relative displacement between the pin and plate. Whenever the interfacial stress tends to exceed $\mu\sigma_r$, slip occurs locally along the interface.Prior to loading, the interference λ puts the plate in a state of uniform pure radial compressive stress and zero tangential shear stress. When the plate is loaded ,shear

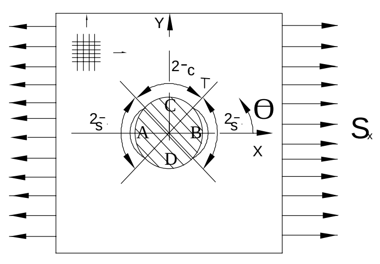

Fig.1Configuration with uniaxial plate load

stress are developed along the interface and the normal stresses under go redistribution. The stress distribution at the non slip stage corresponds to that of an interference fit pin with a perfectly rough interface. Slip is initiated when and at the location where the maximum value of the $|\sigma_{r\theta}| / |\sigma_r|$ on the interface just exceeds the interfacial friction co-efficient μ.

ANALYSIS

The plate is subjected to uniaxial tension.The center of the pin coincides with the origin of the co-ordinate system.Cartesian co-ordinates are used for the outer boundary and cylindrical co-ordinates for the hole boundary.Orthotropic plate is charecterised by the four elastic constants E_1,E_2,G_{12} and υ_{12}.Since the problem is double axis symmetric,only quarter domain is analysed as shown in fig.2.

The boundary conditions on the hole boundary with infinite friction are

a. $\sigma_r = 0$, $\sigma_{r\theta} = 0$ in the region of separation

b. $U = a\lambda$ and Tangential displacement $V = 0$ in the region of contact

With finite friction ,boundary conditions on the hole boundary are

$U= a\lambda$ and $V = 0$, in the region of contact

$t_1 > \mu t_2$ in the region of slip,where t1 and t2 are tangential and normal tractions

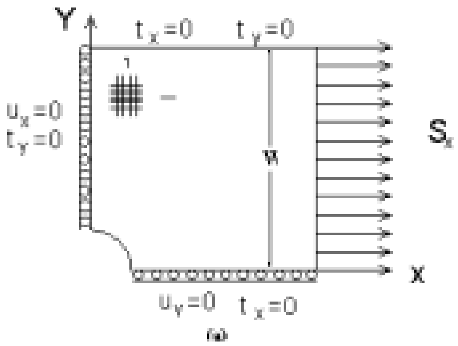

Fig.2(a) Quarter domain Boundary conditions

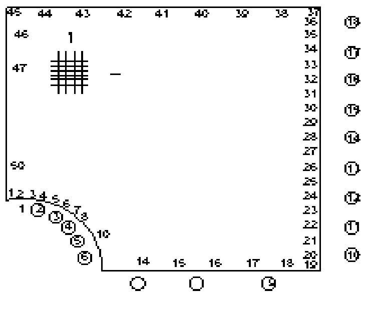

Fig. 2(b) Quarter domain with discretisation

For analyzing a problem, at every node 2 of the 4 variables ux, uy, tx, ty, must be known. At the hole boundary displacements and tractions are transformed into Cartesian displacements as follows.

$$t_1 = -t_x \sin\alpha + t_y \cos\alpha$$

$$t_2 = -t_x \cos\alpha + t_y \sin\alpha$$

But $t_1 = \mu t_2$

Therefore, $t_x = t_y \left[\dfrac{\cos\alpha - \mu\sin\alpha}{\mu\cos\alpha + \sin\alpha}\right]$

$$U_y = \frac{U - u_x \cos\alpha}{\sin\alpha}$$

Substituting these values in the equation of submatrices of the equation [A][u] = [B][t]

$$\begin{bmatrix} A11 & A12 \\ A21 & A22 \end{bmatrix} \begin{bmatrix} \dfrac{U - u_x \cos\alpha}{\sin\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} B11 & B12 \\ B21 & B22 \end{bmatrix} \begin{bmatrix} t_y \left(\dfrac{\cos\alpha - \mu\sin\alpha}{\mu\cos\alpha_{t_y} + \sin\alpha} \right) \end{bmatrix}$$

Bringing the unknowns to the left side

$$\begin{bmatrix} A11 - A12\left(\dfrac{\cos\alpha}{\sin\alpha}\right) - \left[B11\left(\dfrac{\cos\alpha - \mu\sin\alpha}{\mu\cos\alpha + \sin\alpha}\right) + B12\right] \\ A21 - A22\left(\dfrac{\cos\alpha}{\sin\alpha}\right) - \left[B21\left(\dfrac{\cos\alpha - \mu\sin\alpha}{\mu\cos\alpha + \sin\alpha}\right) + B22\right] \end{bmatrix} \begin{bmatrix} u_x \\ t_y \end{bmatrix}$$

$$= \begin{bmatrix} -A12\left(\dfrac{U}{\sin\alpha}\right) \\ -A22\left(\dfrac{U}{\sin\alpha}\right) \end{bmatrix}$$

From this u_x and t_y are evaluated.

The following steps are followed to get the solutions for the problem.

1. Discretize the quarter domain only on the boundary such that the transition point between the contact and separation region is a node. The nodes are at equidistant on the outer boundary of the plate on which the uniformly distributed load is assumed. Boundary element discretization is shown in fig.2b.

2. BEM solution is obtained for 2 different cases of loading

a. $S_x = 0$

b. $S_x = S_{x1}$

Let ‘ T ’ be the slip point at which load required for initiation of slip is to be determined.

Let $F_1 = t_{1a} - \mu t_{2a}$ and $F_2 = t_{1b} - \mu t_{2b}$ for two different cases of loading.At node ‘ T ’ radial force

$F_t = t_1 - \mu t_2 = A\lambda + BS_x$

When ‘ S_x ’ equal to zero,$A = F_1/\lambda$

When S_x equal to S_{x1},$F_2 = A\lambda + BS_{x1}$

Therefore, $$B = \frac{F_2 - F_1}{S_{x1}}$$

At the node ' T ' where slip is initiated
$F_t = t_1 - \mu t_2 = 0$,therefore $A\lambda + Bs_x = 0$

Substituting the values of A and B in the equation we get

$$S_x = \frac{F_1 S_{x1}}{F_1 - F_2}$$

From this value, the values of tractions, stresses and displacements are obtained by interpolation.

RESULTS AND VALIDATION

The Boundary element software is developed for pin joints with finite friction in this section. Therefore, numerical results are obtained for the configuration for which results are available. Boron Epoxy with 60% epoxy by volume having the following properties is used.
E_1=159.1711 Gpa (23.085x10 ^{6}PSI)
E_2=21.8834 Gpa(3.1738x10 ^{6}PSI)
G_{12}=15.2697 Gpa(2.2146x10 ^{6}PSI),ν_{12}=0.2601

Table–1Loads for specified separation angles.Infinite friction.Comparison of BEM and FEM solutions.

Sl. No.	Angle of separation θ_s	$S_x/E_1\lambda$ (x 10^{-6} psi) BEM	$S_x/E_1\lambda$ (x 10^{-6}) FEM (6,7)
1	0.0	6.3	6.627
2	11.25	6.35	6.713
3	22.5	6.66	7.043
4	33.7	7.594	7.733
5	45	9.529	9.215
6	56	15.87	13.330

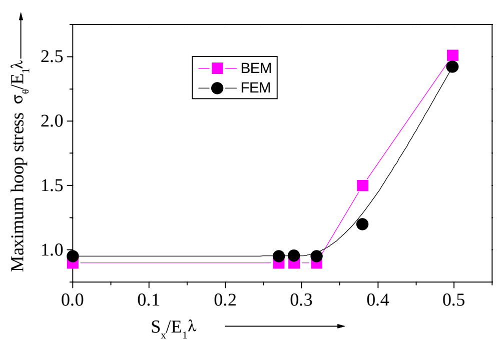

Fig. 3 Comparison of maximum hoop stress around hole boundary with applied plate load. Comparison of BEM and FEM results.

Convergence test were carried out to find the right number of elements.Then solutions were obtained for different configuration .Table 1 shows the variation of causative loads for the specified separation in orthotropic domain and with infinite frictional interface.Fig.3 shows the variation of maximum hoop stress on the hole boundary with applied load in this receeding contact problem These solutions are compared with FEM solutions available in the literature and is found to be in good agreement.The deviations of the boundary conditions are also found to be relatively less .

Table 2 gives the load required for intiation of slip at different location with different co-efficient of friction.The material of the plate is given isotropic values in order to compare the solutions available in the literature by Continuum technique.

CONCLUSIONS

Thus the software under development can be used effectively for contact stress problems and it is checked for double axis symmetric orthotropic plate under plate load and with frictional interface. The problem analysed here is receeding contact problem and three dimensional effect are neglected.. However, the software will be extended to joints with elastic pin and also with clearance fit.

Table –2 Load required for intiation of slip.Finite friction.Comparison with continuum solutions. Isotropic domain.

ANGLE	Co-efficient of friction	$S_x/E\lambda$ BEM	$S_x/E\lambda$ Continuum(10)
43°	0.1	0.113	0.09
39.3°	0.2	0.1619	0.164
37°	0.3	0.224	0.22
32°	0.4	0.2507	0.254
30°	0.5	0.294	0.3
21°	1.0	0.4105	0.42

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